International Journal of
Technology and Emerging Research
ISSN: 3068-109X (online) | ISSN: 3068-1995 (print) | https://www.ioro.org/ijter

Analysis of Differential Equation for Mathematical Modeling of Physical Phenomena

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Abstract

-The Differential equations play a very important role in the mathematical modelling of physical phenomena by explaining how physical quantities differ with respect to space and time. These applications like, heat conduction is one of the most important process modeled using partial differential equations, particularly the heat equation. This study throws light on the analysis of differential equations arising in heat conduction and their numerical treatment using the matrix method. The continuous heat conduction model is discretized into a system of algebraic equations through finite difference approximation, which is then represented in matrix form for accomplished calculation. The matrix method presents an orderly approach for solving large systems of equations and analyzing temperature distribution within conductive media under various boundary and initial conditions. This approach shows that the physical importance of differential equations in engineering and applied sciences while throw light on the effectiveness of matrix techniques in obtaining approximate solutions to heat transfer problems.In this problem we find general solution of heat conduction equation by using matrix method..

Keywords: *Differential equation,Matrix method,Eigen value,Eigen vector,ordinary differential equation,Partial differential equation.*

1. Introduction

Differential equations serve as the foundation of mathematical modeling, providing an effective mechanism to describe and examine physical phenomena that implies a major change in nature.Most natural phenomena such as motion, heat transfer, fluid flow, population growth, and electrical circuits depends on how quantities vary with respect to time or space. Differential equation provides a strict mathematical structure to model how quantities change. An analysis of heat conduction is one of the initially and most significant example of how the differential equation are used to model physical phenomena. It give a clear delineation of how mathematics put real-world processes into specific analytical form [1]. Heat conduction is frequently characterized by differential equation to predict temperature distributions in physical systems [2]. Other researchers have solved the heat conduction equation by using separation of variables. However the recency of this study reclines in the use of the matrix method as another approach to obtain the general solution. Instead the separation of variables method, the matrix based proposal combined with graphical analysis not only simplifies the general solution but also upgrade the physical evaluation. The study of heat conduction with a heat source integrates mathematical theory and practical applications, providing insightful information and useful tools for comprehending and forecasting the behaviour of physical systems. Scientists and engineers can better comprehend heat conduction phenomena and come up with efficient solutions for practical issues by creating mathematical models based on these ideas[3].Conduction is the transfer of energy from the more energetic particles of a substance to the adjacent less energetic ones as a result of interaction between the particles[4].The use of the heat use of the heat equation with heat sources has applicability in a number of areas,including heat transmission, transportation issues, and hydrology[5].The additional terms in the heat equation can be bought of as the source term denoting a chemical process in the context of biology[6].The study of heat conduction with a heat source integrates mathematical theory and practical applications, providing insightful information and useful tools for comprehending and forecasting the behaviour of physical systems[7].Scientists and engineers can better comprehend heat conduction phenomena and come up with efficient solutions for practical issues by creating mathematical models based on these ideas[8].

2. Mathematical Modeling

The temperature distribution in a rod is governed by the heat conduction equation:

$$\frac{\partial u}{\partial t}=\alpha\frac{\partial^2 u}{\partial x^2} \quad (1)$$

• $u(x,t)$= temperature at position x and time t

• α =thermal diffusivity

3. ≤x ≤L, t>0 (2)

In equation (1), $\frac{\partial u}{\partial t}$denotes the partial derivative of temperature u with respect to time t, andα denotes the material's thermal diffusivity, and $\frac{\partial^2 u}{\partial x^2}$denotes the second partial derivative with respect to space$x$.

4. Initial Condition

The initial condition indicates the temperature distribution at time t = 0:

• $u(x,t)=f(x)$

This represents the distribution of heat along the rod initially.

5. Boundary Condition

Boundary condition explains the behavior of temperature at the ends of the rod.

6. Dirichlet Boundary Condition (fixed poit)

• $u(x,t)$ = 0,$u(L,t)$= 0

This represents the both ends of the rod are retain at zero temperature.

7. Neumann Boundary Condition (insulated ends)

$$\frac{\partial u}{\partial x}(0,t)=0,\frac{\partial u}{\partial x}(L,t)=0$$

This means heat is not flowing through the end point of rod.

8. Mixed Boundary Condition

In this condition one end is fixed, and the other is insulated.

9. How Heat is transferred in the system

Heat transfer is the process at which the heat energy flowing between part of a system due to temperature distribution. According to second law of thermodynamics. Heat always flows from higher temperature to lower temperature. In a solid heat moves through by conduction. For Example:

- Heat transfer in a metal rod.
- Heat moving through a wall.

10. Discretizations

If we divide the domain L (rod) into N equally spaced nodes: $x_1, x_2, x_3, \ldots\ldots, x_n$

Now by using the finite difference approximation: Let us consider u (x) be a function and Δx be a small step size.

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2}$$

Where u_{i-1} represents the node to the left and u_{i+1} denotes the node to the right and u_i represents the node itself and $\frac{u_{i+1} - u_i}{\Delta x}$ denotes the right slope and is difference between i and its right neighbor and $\frac{u_i - u_{i-1}}{\Delta x}$ denotes the left slope and is difference between i and its right neighbor.

$$curvature \approx \frac{\text{right slope} - \text{left slope}}{(\Delta x)}$$

$$\approx \frac{\frac{u_{i+1} - u_i}{\Delta x} - \frac{u_i - u_{i-1}}{\Delta x}}{\Delta x}$$

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2}$$

and the heat equation can be written as:-

$$\frac{du_i}{dt} = \alpha \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2}$$

and the system can be written as:

$$\frac{dU}{dt} = AU$$

where U (t) $= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ . \\ . \\ . \\ u_n \end{bmatrix}$

and

A =

$$\frac{\alpha}{(\Delta x)^2}\begin{bmatrix} -2 & 1 & 0 & 0 & \dots & \dots \\ 1 & -2 & 1 & 0 & \dots & \dots \\ 0 & 1 & -2 & 1 & \dots & \dots \\ 0 & 0 & 1 & -2 & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Now the eigen value of A is:-

A =

$$\frac{\alpha}{(\Delta x)^2}\begin{bmatrix} -2 & 1 & 0 & 0 & \dots & \dots \\ 1 & -2 & 1 & 0 & \dots & \dots \\ 0 & 1 & -2 & 1 & \dots & \dots \\ 0 & 0 & 1 & -2 & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Let us represent the eigen value of A by λ_n,we have:

$$\lambda_n = \frac{\alpha}{(\Delta x)^2}\left(-2+2\cos\left(\frac{n\pi}{N+1}\right)\right); n=1,2,3,4,\dots..,N$$

Using identity:

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$\lambda_n = \frac{\alpha}{(\Delta x)^2}\left[-4\sin^2\left(\frac{n\pi}{2(N+1)}\right)\right]$$

And

$$\Delta x = \frac{L}{N+1} \rightarrow N+1 = \frac{L}{\Delta x}$$

$$\lambda_n = \frac{\alpha}{(\Delta x)^2}\left[-4\sin^2\left(\frac{n\pi(\Delta x)}{2L}\right)\right]$$

$$\sin^2\left(\frac{n\pi(\Delta x)}{2L}\right) \approx \left(\frac{n\pi(\Delta x)}{2L}\right)^2$$

$$\lambda_n \rightarrow \frac{-4\alpha}{(\Delta x)^2}\frac{n^2\pi^2(\Delta x)^2}{4L^2}$$

$$\lambda_n = \frac{-\alpha n^2\pi^2}{L^2}$$

Now let us find the eigen vector:

Let us consider the difference equation:

$$u_{i+1} - 2u_i + u_{i-1} = \lambda u_i$$

Now, rearranging the above equation , we get:

$$u_{i+1} - (2+\lambda)u_i + u_{i-1} = 0$$

Because the above equation is second order difference equation with zero boundaries$(u_0 = 0)$, Let us suppose a solution involving a sine function.

$$u_i = C\sin i\theta$$

Now let us take C = 1 for simplicity

$$u_i = \sin i\theta$$

$$\sin(i-1)\theta - (2+\lambda)\sin(i\theta) + \sin(i+1)\theta = 0$$

Using the identity:

$$\sin(A+B) + ¿\sin(A-B) = 2\sin A\cos B¿$$

Now for the non-trivial solution:-

$$2\cos\theta - (2+\lambda) = 0$$

$$2\cos\theta = (2+\lambda)$$

$$\lambda = -2(1-\cos\theta)$$

$$\lambda = -2(2\sin^2\frac{\theta}{2})$$

$$\lambda = -4\sin^2\frac{\theta}{2}$$

Apply the boundary conditions: $u_o = 0, u_{N+1} = 0 \wedge \mathit{using\ the\ solution}\, u_i = \sin i\theta$

And taking i = N + 1

$$\sin(N+1)\theta = 0$$

$$(N+1)\theta = n\pi$$

$$\theta_n=\frac{n\pi}{N+1}\,;\,where\,n=1,2,3,4,\ldots\ldots,N$$

$$u_i^n=\sin\left(i\frac{n\pi}{N+1}\right)$$

or

$$u_n=\sin\left(i\frac{n\pi}{N+1}\right)$$

The general solution ;

$$u_i(t)=\sum_{n=1}^{N}C_n u_n e^{\lambda_n t}$$

$$u_i(t)=\sum_{n=1}^{N}C_n\sin\left(i\frac{n\pi}{N+1}\right)e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t}$$

$$u_i(t)=\sum_{n=1}^{N}C_n\sin\left(\frac{n\pi x}{L}\right)e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t}$$

This is required general solution of the heat conduction.

11. Result Analysis

The heat conduction problem has been analyzed by using the matrix method. The temperature distribution is explained by the combination of eigen values, eigen functions and time-dependent coefficients.

The general solution of the heat conduction:

$$u_i(t)=\sum_{n=1}^{N}C_n\sin\left(\frac{n\pi x}{L}\right)e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t}$$

The space part of the solution and each mode looks like a sine curve.

$$y=\sin\left(\frac{n\pi x}{L}\right)$$

This means that if

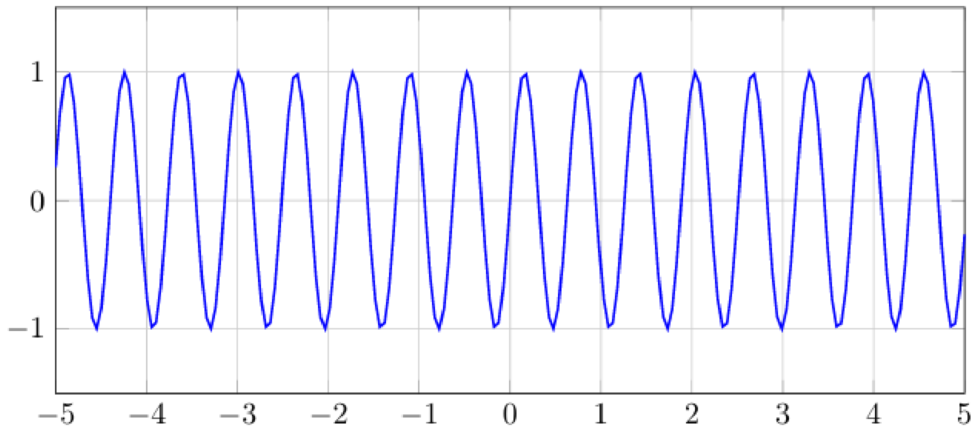
n = 1 → one smooth hump

n = 2 → two humps

n = 3 → three humps

higher

n → more oscillation



Now the time effect:- Each mode has a time part that is $e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t}$ and this is called decay factor.

At t = 0

$e^0 = 1$ it represents no decay mean value does not decrease with time.

At t >0 exponential becomes smaller, and then the temperature starts decreasing with time.

As t →∞then the temperature becomes zero. We can also say the temperature become uniform.

12. Conclusion

The matrix method provides an efficient and systematic approach for analyzing heat conduction problems governed by differential equations. By transforming the heat conduction equation into matrix form. The eigenvalue–eigenvector formulation of the matrix system helps determine the temperature distribution and thermal behavior of the conducting medium accurately. Therefore, the matrix method is an effective mathematical tool for studying heat conduction phenomena in engineering and physical applications. In this study, the general solution of the heat conduction equation has been obtained using the matrix method. The matrix method simplifies the derivation of the general solution and offers an organized technique for analyzing heat transfer problems in conducting media. Hence, the method demonstrates strong applicability for solving heat conduction models arising in mathematical physics and engineering.

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