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A Comparative Study of Inapplicable cases And Limitations in Methods for Solving Second Order Differential Equations with Constant Coefficients

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Abstract

In this paper we compare the inapplicable cases of the Method of Undetermined Coefficients and Method of Variation of Parameters in solving second order differential equations with constant coefficients. We first analyse the cases of inapplicability for each method to identify the inapplicability, investigate their respective limitations and the underlying reasons responsible for it .The applicability of Method of undetermined coefficients clearly depends upon the forcing functions on the other hand the method of variation of parameter limits itself over computational complexities over non elementary integrals. Then finally we carried out comparative analysis to highlight the differences and sameness between the limitation of these methods.

Keywords: Homogeneous , Non homogeneous , Method of undetermined coefficients , method of variation of parameters.

1. Introduction

An equation involving derivatives of one or more dependent variable with respect to one or more independent variables is called differential equation[1].The second order linear differential equations with constant coefficient which is non homogeneous have the general form: $a\,y' + b\,y'' + cy = f(x)$ where a, b, c are constants and $f(x)$ is any non zero function . A differential equation with constant coefficients is said to be the one when the dependent variable and its derivatives occur in it in the first degree and are not multiplied together[2].When f(x) becomes zero then it becomes the corresponding homogeneous form of the non homogeneous equation .It has several applications in real world phenomena across physics , engineering, and economics .The comparative study of analytical methods as well as analytic and numeric methods are already has been discussed but this paper focuses on the edge or failure cases and the limitations of the two methods that is method of undetermined coefficients and variation of parameter.

2. Methodology

In this section , first we discuss the working method of both the methods that is the method of undetermined coefficients and variation of parameters. The general form of second order linear differential equation with constant coefficients is given by: $a\,y' + b\,y'' + cy = f(x)$

The general solution is always the sum of complementary solution(obtained by the corresponding homogeneous part of the given non homogeneous equation) and particular solution(on which the method primarily focuses) **For Complementary Solution:** Solve the corresponding homogeneous equation

$a\,y^{'}+b\,y^{''}+cy=0$ by using characteristic equation method, which is foundational method for both the methods.

Method Of Undetermined coefficients:

We formulate the initial guess depending upon the nature of the forcing function .Compare the complementary solution with initial guess, if it do not overlaps with complementary solution then initial guess becomes the particular solution. But if it overlaps with complementary multiply it with suitable power of x to establish linear independence. Substitute the modified initial guess solution into the given differential equation. Equating the coefficients of like term leads to an algebraic set of equation that enables to obtain the unknown constant and hence the particular solution.

Method of Variation of Parameters:

Variation of parameters is a general method for finding the particular solution of non-homogeneous linear differential equations with constant coefficients. This method can be applied to non-homogeneous equations even when the forcing function is not simple enough to apply the undetermined coefficient method.

Assume that $y_1(x)$ and $y_2(x)$ are the two linearly independent solutions of the corresponding homogeneous equation. The particular solution of the non-homogeneous differential equation is written as

$$y_p=u_1(x)\,y_1(x)+u_2(x)\,y_2(x)$$

To simplify calculations, the auxiliary equations:$u_1^{'}\,y_1+u_2^{'}\,y_2=0$ and $u_1^{'}\,y_1^{'}+u_2^{'}\,y_2^{'}=f\,(x)$are used. They form a system of simultaneous linear algebraic equations in$u_1^{'}$and $u_2^{'}$.Solving the above system yields $u_1^{'}$ and$u_2^{'}$. Using the Wronskian determinant,

$$W(y_1\,,\,y_2)=\begin{vmatrix} y_1 & y_2 \\ y_1^{'} & y_2^{'} \end{vmatrix}$$

the expressions for $u_1\,'$ and $u_2^{'}$ are derived. Then, the resulting expressions are integrated to find $u_1(x)$ and $u_2(x)$.

Substituting the expressions for u_1 and u_2 in the assumed solution formula gives the particular solution. Hence, this method involves finding integrals.

In conclusion, this is a general approach to finding the particular solution of non-homogeneous differential equations; however, it involves complex integrals.

3. Analysis of inapplicability and computational limitations

Failure cases of method of undetermined coefficients

Applicability of this method only depends upon the nature of the forcing functions which means it is applicable to a set of functions whose derivate is same as the function itself as polynomials , exponential functions and trigonometric functions. This method is not applicable to all trigonometric functions but only to $sinx$ and $cosx$ .To show the limitations of the method ,consider the case where the forcing function, $f(x)=tanx$. Following are the reasons possible due to which the method limits itself for $tanx$:

• $tan\,x$ in terms of $sin\,x$ and $cos\,x$

As the method of undetermined coefficients is applicable for the forcing function which involves $sin\,x$ and $cos\,x$ so we can write$tanx = sinx$ ¿. But it voilates the condition that the forcing function should belongs to a finite dimensional family of functions and also derivatives belong to the same family.

For example ,forcing functions like $sine, cosine$, polynomials, and exponential satisfy this property and the method of undetermined coefficients is applicable on these forcing functions.

• Non closure under differentiation

The derivatives of $tan\,x$ produces infinitely many forms and it does not belong the same family to $tan\,x$ belongs. But in case of $sinx$ and $cosx$ this is not the case.

$$\frac{d}{dx}(sinx) = \frac{cosx \wedge d}{dx}(cosx) = -sinx$$

which can be written as finite linear combination. But in case of $tan\,x$, this is not possible, because

$$\frac{d}{dx}(tanx) = sec^2 x\ ,\ \frac{d^2}{dx^2}(tanx) = 2\,sec^2 xtanx$$

and so on.

With the successive derivatives ,which involves the powers of $tanx$and $secx$. Hence, $tanx$ cannot be expressed as linear combination of its derivatives.

• Structural Difference

If we keep in mind the perspective of power series expansions , functions like $e^x, cosx$ and $sinx$ have maclaurin series that converge for all real values and show predictable derivative behavior . But in contrast, series of $tanx$has a limited radius of convergence and leads to more complex derivatives.

• Singularities

There is no singularities in $e^x$and $sinx$ which is clear from their respective series:

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots\ldots..$$

$$sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots..$$

$$tanx = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots\ldots$$

The maclaurin series converges of $tanx$ for $|x| < \frac{\pi}{2}$. The series has finite radius of convergence and derivatives generate increasingly complex expressions.

Failure cases of method of variation of parameters

• Dependence on wronskian

This method needs the wronskian of the complementary solution and the wronskian should be non-zero and if the wronskian vanishes then the method fails completely as variation of parameters require linearly independent solutions. But it always works for linear differential equation with constant coefficients. For linear differential equation with variable coefficients, singularities can appear and in those cases wronskian vanishes and the method fails completely.

For example: $x^2 y'' - x y' + y = 0$ The complementary solution is $y_1 = x, y_2 = xlnx$

W(y_1, y_2) = $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

W(y_1 ,y_2) = $\begin{vmatrix} x & x \\ 1 & 1 \end{vmatrix}$

W(y_1 ,y_2) = x·1 −x·1 = 0

We can see that here the wronskian vanishes and hence the method of variation of parameters become inapplicable in case of linear differential equation with variable coefficients.

• ***Complexity of integration***

Method of variation of parameter involves integration as an important step, and sometimes the integration becomes complicated. The integral requires more time and the computational efficiency is less. This method does not fails technically but surely has limitation as compared to other methods.

For example, $y'' - y = x$, the complementary solution is $y_1 = e^x, y_2 = e^{-x}$. Now integrals for method of variation of parameters;

W(y_1, y_2) = $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

W(y_1 , y_2) = $\begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$

W(y_1 , y_2) = -2 ≠ 0 and hence, $y_p = \left(\frac{e^x}{2}\right) \int x e^{-x} dx + \left(e^x ¿\right) \int x e^x dx$

which is solvable but again it will take more time to solve as integration involves the method of integration by parts.

• ***Dependence on complementary solution***

Finding complementary solution is the first step to find the complete solution of the linear differential equation with constant coefficients. If the corresponding homogeneous equation cannot be solved then we cannot even start the method of variation of parameters as this method lacks the initial structure.

• ***Case of non-elementary integrals***

Method of variation works almost for every function but sometimes it provides a solution with non elementary integral such as e^{x2} error function is not elementary.

For example: $y'' + y = e^{-x2}$,then the complementary solutions are $y_1 = cosx, y_2 = sinx$ and W=1.

$u_1 = \int -sinx\, e^{x2} dx$ & $u_2 = \int cosx\, e^{-x2} dx$

On solving these integrals, integrals cannot be expressed in elementary form but give

rise to the error function $erf(x)$.

• *Failure in non-linear differential equations*

This method is only applicable for linear differential equations as it depends upon the linear combination of complementary solution and the superposition principle does not hold for non linear differential equations and thus the method fails for non linear equations.

For example: y´´ +y2 = x, here y2 represents the non linear term and hence there is no complementary solution . Thus variation of parameters cannot be applied to it.

4. Comparison of limitations of Method of undetermined coefficients and Method of variation of parameters

Method of undetermined coefficients and method of variation of parameters differs from each other in many aspects but both the methods limits itself at one common point that is dependency on complementary solution to get the complete solution. Following are the few aspects in which both the methods differs from each other which helps in the applicability of both the methods in different situations:

Aspect	Method of undetermined coefficients	Method of variation of parameters
Forcing Function	Applicable to some limited functions such as e^{ax}, polynomials, $sinx$, $cosx$.	There is no restrictions on forcing functions.
Computation	No computational complexity as there is no integration involved.	Computational complexity arises due to integrals which cannot expressed as elementary functions.
Generality	No computational complexity as there is no integration involved.	It can be applied to linear differential equations with constant and variable coefficients.
Equation type	Linear differential equation with constant coefficients only.	It can be applicable to linear equations with variable coefficients.
Wronskian	No requirement of wronskian.	Finding a nonzero wronskian is an efficient step.
Solution Assumption	It requires an initial guess to form solution and hence for complete solution.	No assumed initial guess is required.

5. Conclusion

In this paper, we investigated the limitations and inapplicability cases of the Method of undetermined coefficients and variation of parameters. Both the methods have dependence on characteristic equation method for the complementary solution. This analysis shows that method of undetermined coefficients is computationally more efficient than Method of variation of parameters but has a key limitation depending upon the forcing function. On the other hand, Method of variation of parameter provides a more general approach and applicable to wider ranges of the function accompanied with a computational complexity for some integrals that may be complicated or not directly solvable in simpler terms. Thus, neither method is superior to each other, their applicability depends on the need situation. Instead their usefulness depends upon the specific features of the differential equation. The comparative perspective clarifies the functional limits of the two methods but creates a thoughtful approach to solve linear differential equations.

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